Note: All angles are in radians.

38. Consider the function $f(x) = \sin(x)$ at $x = 1$.
    (a) What is the exact (symbolic) derivative $f'(x)$ of $f(x)$? Using a computer, evaluate the symbolic $f'(x)$ numerically at $x = 1$.
    (b) Using a computer, evaluate the forward difference quotient $\frac{f(x+h)-f(x)}{h}$, the backward difference quotient $\frac{f(x)-f(x-h)}{h}$ and the central difference quotient $\frac{f(x+h)-f(x-h)}{2h}$ for $h = 1$, $h = 0.1$, $h = 0.01$, $\ldots$, $h = 10^{-16}$.
    (c) Does a smaller h necessarily lead to a more accurate result? Why?
    (d) Which of the difference quotients produces the best results?

39. Implement the **extrapolation method for numerical differentiation** (compute:
    (a) the forward difference quotient,
    (b) the central difference quotient

    at several small values of h and extrapolate to $h = 0$) in MATLAB or Octave. Try out both variants of the algorithm on example 38.

40. Using the formulas for first-order **differential numbers** (f, f') and a computer, compute the derivative of $f(x) = x^2 - 7\sqrt{\sin(x) + xe^{2x+1}}$ at $x = 1$.
    Note: Automating this procedure is called **(forward) automatic differentiation**.

41. Compute the derivative from example 40 using your algorithms from example 39 and compare the computation effort and the accuracy of the two methods (the extrapolation method vs. automatic differentiation using differential numbers).

42. Let $f : \mathbb{R}^n \to \mathbb{R}$.
    (a) How often do you need to evaluate f to approximate its gradient f' using forward difference quotients (with a single fixed h)?
    (b) How often do you need to evaluate f to approximate its gradient f' using backward difference quotients (with a single fixed h)?
    (c) How often do you need to evaluate f to approximate its gradient f' using central difference quotients (with a single fixed h)?
    (d) How often do you need to evaluate f to approximate its gradient f' using an extrapolation method requiring a forward difference quotient for k different values of h?
    (e) How often do you need to evaluate f to approximate its gradient f' using an extrapolation method requiring a central difference quotient for k different values of h?
    (f) How many real components does a differential number $df = (f, f')$ now have? What follows for the effort of automatic differentiation using differential numbers compared to the effort of evaluating only f?
    (g) How do these results generalize to computing the Jacobian g' of $g : \mathbb{R}^n \to \mathbb{R}^m$?

43. Consider the function $f(x) = x\sin(x)$ at $x = \frac{18014}{1625}$.

   (a) What is the exact (symbolic) derivative $f'(x)$ of $f(x)$? Using a computer, evaluate the symbolic $f'(x)$ numerically at $x = \frac{18014}{1625}$.

   (b) Using a computer, evaluate the forward difference quotient and the central difference quotient of f for $h = 1$, $h = 0.1$, $h = 0.01$, $\ldots$, $h = 10^{-16}$.

   (c) Does the forward difference quotient give satisfactory results? The central difference quotient? Why?

   (d) Try your algorithms from example 39 on this example. Do the forward difference quotients now give satisfactory results? Why?

   (e) Compute $f'(\frac{18014}{1625})$ using the formulas for first-order differential numbers (f, f') and a computer.

44. Consider the function $f(x) = \sin(x) + x$ at $x = \frac{355}{113}$.

   (a) What is the exact (symbolic) derivative $f'(x)$ of $f(x)$? What problem do you run into when evaluating this expression numerically at $x = \frac{355}{113}$ with a computer?

   (b) Compute (by hand) a Taylor expansion $t_5(x)$ of order 5 (i.e. up to the $(x-\pi)^5$ term) without the remainder term for $f(x)$ around $x = \pi$.

   (c) Compute (by hand) the derivative $u(x) = t_5'(x)$ of $t_5(x)$.

   (d) Evaluate $u(\frac{355}{113})$ using a computer to obtain an approximation of $f'(\frac{355}{113})$. Why can we neglect the higher-order terms of the Taylor series of $f'(x)$ in double-precision arithmetic?

   (e) Try your algorithms from example 39 on this example. Do they give satisfactory results?

   (f) Compute $f'(\frac{355}{113})$ using the formulas for first-order differential numbers (f, f') and a computer. What can you say about the result?